Curriculum Council

EXAMINERS' REPORT ON 2001 TERTIARY ENTRANCE EXAMINATION

SUBJECT: CALCULUS

STATISTICS

Year	Number Who Sat	Non-Examination Candidates	Did Not Sit
2001	1761	8	61
2000	1886	17	83
1999	1957	7	94

The Examiners' Report is written by the Chief Examiner (or another examiner on their behalf) to comment on matters relating to the Tertiary Entrance Examination in their subject. The opinions and recommendations expressed in this report are those of the Chief Examiner and not necessarily representative of or endorsed by the Curriculum Council.

The Marking Guide provided at the end of this report was prepared for markers and substantially amplified by discussions held in the pre-marking meeting. It is not intended as a set of model answers, and is not exhaustive as regards alternative answers. Some of the answers are less than perfect, but represent a standard of response that the examiners deemed sufficient to earn full marks. Teachers who use this guide should do so with its original purpose in mind.

SUMMARY/ABSTRACT

The general consensus of the markers showed that the paper was well received. There were no signs that candidates had real problems completing it. The paper was well structured while its content was challenging for the candidates without being impossible. One of the markers mentioned that during the marking process it became obvious that the candidates, on average, liked the paper. The mean of 56.8% was close to the Curriculum Council's target of 57%-58%.

There were a few concepts that received special attention in this paper. They were: differentiation from first principles (Question 2b), complex numbers to be shown as directed line segments (Question 7 and 17), differentiation of complex numbers (Question 15), the opportunity to use the dot product (Question 17b) and showing that a function is positive on a given interval (Question 18a). These concepts are clearly described in the syllabus while proper understanding and knowledge of Year 11 concepts should form a sound basis for the Year 12 Calculus course.

This year's paper was the second 'write-on paper' for Calculus. Markers indicated that this format was very beneficial while marking the candidates' work. The only disadvantage was that candidates were not able to take the questions home. The paper contained the required range of skills: rote learning, algebraic and manipulative techniques and problem solving, while a sound knowledge of calculus was required to complete the paper satisfactorily. There were opportunities to make appropriate use of the graphics calculator. One of the markers commented that "the use of the graphics calculator seemed very natural with this paper".

GENERAL COMMENTS

Most markers mentioned that the paper was very fair and well structured. However, the paper was somewhat unconventional in that there was not a uniform flow from easy to hard questions. It became apparent that many candidates either became 'bogged down' in the algebraic intricacies of Question 11 or did not attempt this question at all. Question 13 seemed to have caused some difficulty while Question 20 was well done by the majority of candidates. Overall, this mix of questions was not interpreted as having been detrimental to the candidates. It became evident that there was sufficient material in most exercises to enable candidates to make a reasonable start.

Most of the routine questions were well done although unnecessary manipulative errors were made at the start (Question 1). Some markers advised that candidates should be taught the strategy to recheck the first exercises because 'exam nerves often cause errors here'. It also became apparent that many candidates could handle the technical aspects of calculus well but were not mathematically flexible enough to apply calculus to practical situations: multiple solutions (Question 10), minimum distance (Question 13c, 19b), rate of change (Question 14) and simple harmonic motion (Question 21b).

The statistics at the beginning of this Report show a worrying drop in the number of candidates sitting for their Calculus final examination – a drop of 10% over the past two years. Signals from tertiary institutions indicate that too many students hope to study science or engineering without a background of Year 12 Calculus. A concerted effort is required to promote the importance of this subject to high school students.

The various markers' reports emphasised that specific attention should be paid to the following elements:

(a) Improve students' algebraic, manipulation and simplification skills by means of constant revision.
(b) Improve students' ability to present a mathematical proof by deduction (Question 8b, 18a,b).
(c) Improve students' ability to use a graphics calculator effectively and efficiently.
(d) Improve students' ability to call upon a variety of strategies for problem solving.
(e) One of the markers stated: "If we are to be competitive internationally in IT, engineering and science, then mathematics must be given a higher priority in the general community".

Points for Consideration by Teachers and Students

As has been mentioned, more attention should be paid to revision of material done in Year 11 that forms the basis of studying the many mathematical concepts taught in Year 12. Candidates should also improve in their skills to draw clear sketches of curves and vectors. The use of certain mathematical tools should be considered as a requirement for a final Calculus examination.

COMMENTS ON SPECIFIC QUESTIONS

Question 1 Mean 5.39 out of 7

This was a routine question that was answered well by most candidates. A common error in (a) was (due to exam nerves?) that when $\sqrt{a} = 6 \Rightarrow a = \sqrt{6}$. In part (b), a number of candidates were unsure about the gradient of the tangent line and were of the opinion that $f'(x)$ was the required gradient. Often the calculator value was given instead of the required exact value.

Question 2 Mean 4.01 out of 6

Many candidates used in answering part (a) the method of dominant powers and obtained the incorrect answer. Another common error was that the answer was given as ∞ instead of $-\infty$. In part (b), many candidates were bogged down in the algebra – expanding $f(x+h)$ and the effect of the negative sign - and gave the correct answer based on straightforward differentiation. The method of first principles differentiation seemed to be forgotten.

Question 3 Mean 3.94 out of 6

The circle was often indicated well though the 'roundness' was not neat at all. A number of markers commented that candidates should be required to take a compass to their exam. Many candidates showed difficulty indicating the direction of the ray and the use of the arrow.

Question 4 Mean 4.90 out of 6

Part (a) was well done. In part (b), many candidates overlooked the requirement to draw a diagram first before the differentiability was to be discussed. An interesting factor was that candidates either drew the diagram for $f(x)$ or $f'(x)$.

Question 5 Mean 8.61 out of 11

This question tested basic skills in differentiation and was well done. Common errors were that in (a) $(\ln x)^2$ was simplified into $2 \ln x$ while in (b) many candidates did not recognise that use had to be made of the Fundamental Theorem of Calculus. The implicit differentiation exercise of (c) was well done. It seems that candidates are well prepared for this kind of question.

Question 6 Mean 4.79 out of 8
This question, in particular part (a), was the first real challenge for most candidates. Though two substitutions were given, most candidates did not know how and when to use them. In particular, the ‘dx’, ‘du’ and ‘dq’ caused many headaches. Basic flaws in integration by substitution were apparent. In part (b) many candidates obtained three marks because the absolute value signs were forgotten while the constant of integration was omitted as well. At the markers’ meeting, it was decided not to penalise the candidates for this omission.

Question 7 Mean 3.86 out of 5
It seemed that candidates could change a complex number from Cartesian form to polar form quite well. However, there are still too many mistakes with respect to the value of the argument. Many candidates did not follow the instructions by graphing a complex number as a ‘directed’ line segment.

Question 8 Mean 8.36 out of 11
This question was a standard exercise where basic insight in continuity was required. The mean of 8.36 out of 11 showed that many candidates fared well. In part (b), a large number of candidates tried to prove that f was an odd function by using one specific value for x. A real lack of developing a formal proof became evident. Part (d) was either completely correct or incorrect (due to having the calculator in degree mode). It was quite remarkable that in part (f) candidates often decided that f was continuous at $x = 0$ because the left hand limit was equal to the right hand limit while findings under part (a) were forgotten.

Question 9 Mean 8.64 out of 13
It became obvious that many candidates were not well prepared for this kind of question. In part (a), only a few candidates were able to obtain the four marks for completing the table, in particular the column for f''. The result was that the graph for f' required in part (c) was not drawn well. The x-intercepts were often shown properly but the rest of the graph was incorrect. In part (d), many candidates did not show a proper understanding of the difference between local and global maxima.

Question 10 Mean 5.15 out of 9
The proper use of the graphics calculator would have helped many candidates to understand the periodic nature of this function. In part (a), some unnecessary errors were made when differentiating the function, e.g. when candidates forgot to include π in the first derivative. In parts (b) and (c), many candidates missed out on the multiple solutions that were required. It became clear that finding the maximum of a rate of change troubled many candidates. Some modifications in the marking scheme helped candidates to obtain a number of valuable marks for this question.

Question 11 Mean 1.94 out of 6
This question proved to be quite difficult for the candidates because real algebraic skills were involved. Some changes in the initial marking scheme gave candidates the opportunity to obtain two or three marks for this exercise. A number of candidates were able to obtain the equation $a^2 - a + b^2 - b = 0$ but were unable to complete the square so that the centre and radius of the circle could be determined. However, a few candidates did obtain the required result.

Question 12 Mean 5.83 out of 8
Overall, this question was well done. A few difficulties arose when in part (b) both components were supposed to be equal to zero while in part (c) only the **i**-component was to be equal to zero.

Question 13 Mean 6.00 out of 12
In this question part (a) was done well. For part (b), the graphics calculator should have been a great help when the two functions were to be observed and the areas enclosed by these two curves. Too many candidates limited themselves to finding the area between x=2 and x=3 while the area between x=0 and x=2 was ignored. In part (c), determining the coordinates of point S was easy, but to find by means of differentiation - or implicit differentiation - the minimum distance was too much for most to justify their conclusion. Here was a clear example that candidates were unable to apply their knowledge in solving a practical problem. It should be noted that, in this part, alternative solutions are possible.

Question 14 Mean 5.69 out of 9
The somewhat complicated function with an exponential component in a bracket caused a ‘headache’ with many candidates. Though most candidates were able to determine the value for q, it became too difficult to find the value for p using the expression “for a long time”. In part (b), the rate of change was not understood so that many assumed that $T(0) = 3$ instead of $T'(0) = 3$. The same error was made in part (c).

Question 15 Mean 4.13 out of 6
In part (b), the word 'hence' was often overlooked so that candidates became bogged down in differentiating trigonometric functions. Others forgot to include the i in the answer. Part (d) was very poorly done. It became clear that some revision of Year 11 Geometry & Trigonometry material in Year 12 is not out of place.

Question 16 Mean 4.46 out of 6
This question was done either very well or poorly. Most candidates could interpret the term 'half-life' well and apply it in finding the value for k. It became apparent that many candidates did not know the difference between decimal places and significant figures (they were not penalised!) In part (b), many candidates must have read "decreased to 97%" instead of "decreased by 97%".

Question 17 Mean 2.32 out of 6
This question really blended the two topics of complex numbers and vectors. Part (a) was well done though greater accuracy in drawing the required vectors should be shown by candidates, e.g. by using a ruler. Part (b) was too difficult for most to obtain full marks because no use was made of the dot product of two vectors.

Question 18 Mean 4.46 out of 13
This was a non-standard question which caused conceptual difficulties in parts (a) and (b). To show that if both functions of the integrand are positive that the definite integral will be greater than zero was beyond many candidates' capabilities. It should be noted that a few candidates were able to present an acceptable proof. Parts (c) and (d) were the source of marks for this exercise – if the calculator was in radian mode.

Question 19 Mean 1.94 out of 7
Most candidates were able to complete part (a) though some manipulative errors were obvious. It was somewhat worrying that in part (b), candidates were not able to develop a 'distance' equation that should be differentiated to find the minimum distance. The success rate in part (d) was higher.

Question 20 Mean 7.53 out of 12
It seemed that many candidates were able to complete this question quite well. It could have been placed earlier in the paper. However, most markers were happy with its position at the end of the paper. As a standard question, most candidates attempted it. In part (a), the differentiation was done well though the interpretation of the 'negative' sign was not always correct. Part (b) caused more problems because many candidates did not recognise the trigonometric components as parametric constants and integrated them as well. Part (c) went well while in part (d) many candidates determined the constants of integration - which should have been done in part (b) – and obtained the proper solution. Some candidates seemed to be confused by the fact that the rocket was fired from a position 100 metres above the ground.

Question 21 Mean 2.67 out of 13
This question proved to be the most difficult and demanding exercise of the paper. It seemed that only a few candidates were able to obtain full marks. In part (a), those candidates who remembered the chain rule in differentiation were able to get full marks for this section. In part (b), the extra information that use could be made of a simple harmonic motion did not deter many candidates to develop linear or exponential equations. It was disappointing that so few candidates were able to use the standard SHM formula involving a phase shift. For that reason, part (c) was done incorrectly as well.

POINTS FOR CONSIDERATION BY THE SYLLABUS COMMITTEE

Nil

George Spyker
December 2001

2001 Examining Panel
Chief Examiner: Dr Geert Spyker
Deputy: Dr Nandita Rath
Third Member: Mr Ian Hailes

Chief Marker: Mr Don Utting

CALCULUS TEE 2001 MARKING GUIDE

1.(a) $\int_1^a x^{-\frac{1}{2}} dx = 10$ ✓ $\Rightarrow \left[2x^{\frac{1}{2}}\right]_1^a = 10$ ✓ $\Rightarrow 2\sqrt{a} - 2 = 10 \Rightarrow a = 36$ ✓ [3 *marks*]

1.(b) $f'(x) = 2\cos x + 2\sin 2x$ ✓

$m = f'(\frac{p}{6}) = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$ ✓

$y = mx + c$

$\frac{1}{2} = 2\sqrt{3}.\frac{p}{6} + c \Rightarrow c = \frac{1}{2} - \frac{p\sqrt{3}}{3}$ ✓

$\therefore y = 2\sqrt{3}x + \frac{1}{2} - \frac{p\sqrt{3}}{3}$ ✓

Alternatively:

$y - y_1 = m(x - x_1)$

$y - \frac{1}{2} = 2\sqrt{3}(x - \frac{p}{6})$ (✓)

[4 *marks*]

[total 7 *marks*]

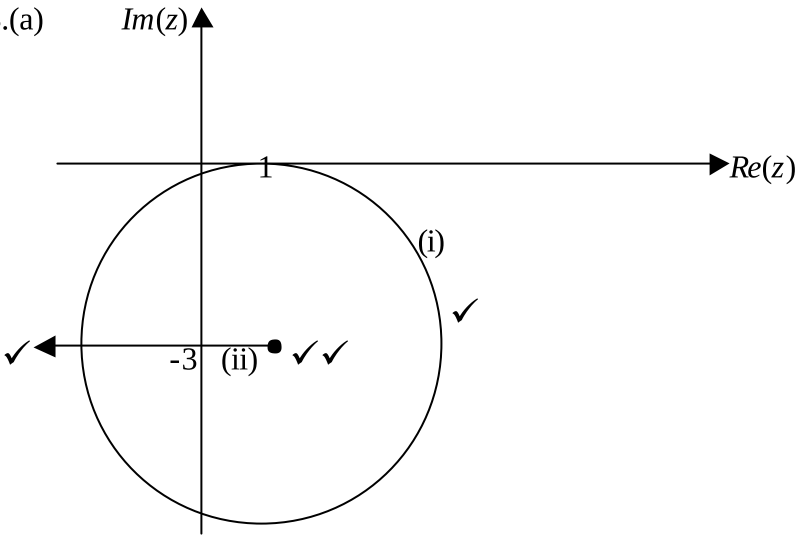
2.(a) $\lim_{x\to\infty} \frac{2x^2 - 3x^3 + 4x - 8}{6x^2 - 5x + 8} = \lim_{x\to\infty} \frac{2 - 3x + \frac{4}{x} - \frac{8}{x^2}}{6 - \frac{5}{x} + \frac{8}{x^2}}$ ✓ $= -\infty$ ✓ [2 *marks*]

2.(b) $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 + 4x + 4h - 3x^2 - 4x}{h}$ ✓✓

$\lim_{h\to 0} \frac{6xh + 3h^2 + 4h}{h}$ ✓ $= \lim_{h\to 0} (6x + 3h + 4) = 6x + 4$ ✓ [4 *marks*]

[total 6 *marks*]

3.(a)



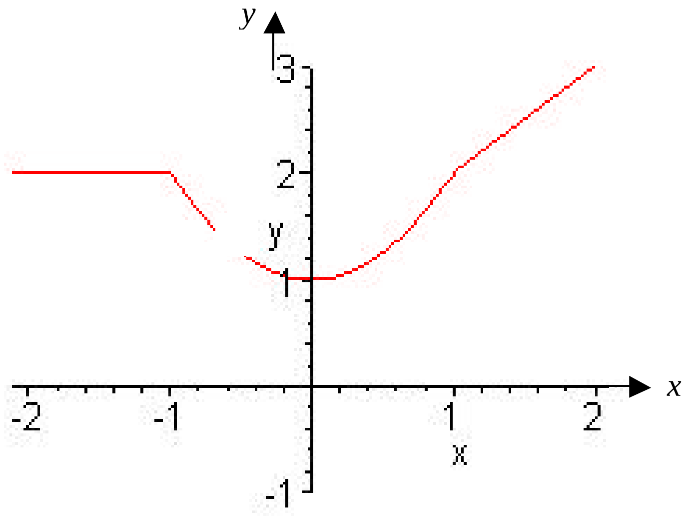
[4 *marks*]

3.(b) $z = -2 - 3i$ ✓✓ [2 *marks*]

[total 6 *marks*]

4.(a) $f(-1) = 2$.
Then $a + 1 = 2 \Rightarrow a = 1$. ✓
$f(1) = 2$.
Then $b + 1 = 2 \Rightarrow b = 1$. ✓ [2 *marks*]

4.(b)



At $x = -1$, there is a clear corner. Therefore, f is not differentiable at that point. ✓✓

Alternatively:
As $x \to -1^-$, $f'(x) = 0$, $\therefore f'(-1^-) \to 0$.
As $x \to -1^+$, $f'(x) = 2x$, $\therefore f'(-1^+) \to -2$.
$\therefore f'(-1^-) \neq f'(-1^+)$ so the derivative does not exist. (✓✓)

As $x \to 1^-$, $f'(x) = 2x$, $\therefore f'(1^-) \to 2$.
As $x \to 1^+$, $f'(x) = 1$, $\therefore f'(1^+) \to 1$. [4 *marks*]
$\therefore f'(1^-) \neq f'(1^+)$ so the derivative does not exist. ✓✓

[total 6 *marks*]

5.(a) $\frac{dy}{dx} = 2(\ln x).\frac{1}{x}$ ✓✓ $- \frac{1}{x^2}.2x$ ✓ $= \frac{2(\ln x - 1)}{x}$ [3 *marks*]

5.(b) Let $u = x^2$ ✓ $\Rightarrow \frac{du}{dx} = 2x$.

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ ✓

Then: $\frac{dy}{dx} = \sqrt{1+u^3}.2x$ ✓ $= 2x\sqrt{1+x^6}$. ✓

NOTE: (-1 ✓ if solution in t).

Alternatively:

$\frac{dy}{dx} = \sqrt{1+(x^2)^3}.2x$ (✓✓✓)

$= 2x\sqrt{1+x^6}$ [4 *marks*]

(✓)

5.(c) $\frac{dy}{dx} = \sin y$ ✓ $+ x \cos y \frac{dy}{dx}$ ✓✓

$\therefore \frac{dy}{dx} = \frac{\sin y}{1 - x\cos y}$ ✓ [4 *marks*]

[total 11 *marks*]

6.(a) Let $u = e^{2x} \Rightarrow \dfrac{du}{dx} = 2\,e^{2x}$. ✓

Then the integral becomes: $\dfrac{1}{2}\displaystyle\int \dfrac{1}{\sqrt{1-u^2}}\,du$ ✓

Let sin ? = $u \Rightarrow \dfrac{du}{dq}$ = cos ?. ✓

Then the integral becomes: $\dfrac{1}{2}\displaystyle\int \dfrac{\cos q}{\sqrt{1-(\sin q)^2}}\,dq \quad = \dfrac{1}{2}\int dq \;=\; \dfrac{1}{2}q + c$ ✓

$= \dfrac{1}{2}\sin^{-1}(u) + c \;=\; \dfrac{1}{2}\sin^{-1}(e^{2x}) + c$. ✓ [5 *marks*]

6.(b) Let $u = x + \cos x \Rightarrow \dfrac{du}{dx} = 1 - \sin x$ ✓ [3 *marks*]

Then the integral becomes: $2\displaystyle\int \dfrac{1}{u}\,du \;= 2\ln|u| + c = 2\ln|x + \cos x| + c$. ✓✓

[total 8 *marks*]

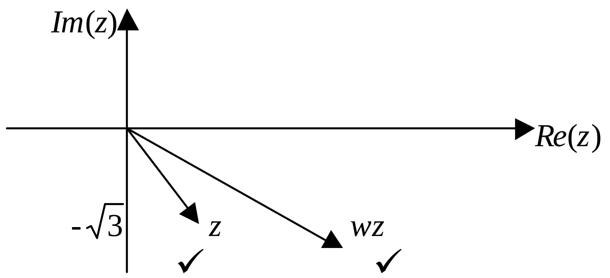
7.(a) $z = 1 - i\sqrt{3} = 2\ \text{cis}\,(-\dfrac{p}{3})$ ✓ and $w = 2\ \text{cis}\,\dfrac{p}{6}$.

Then $wz = 4\ \text{cis}\,(-\dfrac{p}{6})$ ✓✓

Alternatively: $z = 1 - i\sqrt{3}$ and $w = 2\ \text{cis}\,\dfrac{p}{6} = \sqrt{3} + i$. (✓)

Then $wz = 2\sqrt{3} - 2i$. (✓✓) [3 *marks*]

7.(b)



[2 *marks*]

[total 5 *marks*]

8.(a) $x \neq 0$. ✓ [1 *mark*]

8.(b) $f(-x) = \dfrac{1-\cos(-x)}{-x} = -\dfrac{1-\cos x}{x} = -f(x)$ ✓✓

$\therefore f(x)$ is an odd function. [2 *marks*]

8.(c) $\displaystyle\lim_{x\to\infty} \dfrac{1-\cos x}{x} = 0$. ✓✓ [2 *marks*]

8.(d)

x	0.1	$\frac{p}{2}$	$\frac{3p}{2}$	$2p$	$3p$	$5p$
$\frac{1-\cos x}{x}$	0.04996	0.63662	0.21221	0	0.21221	0.12732

✓✓✓

[3 *marks*]

8.(e) $\lim_{x\to 0^-} f(x) = 0$ and $\lim_{x\to 0^+} f(x) = 0$.

$\therefore \lim_{x\to 0} f(x) = 0$. ✓ [1 *mark*]

8.(f) We know that $\lim_{x\to 0} f(x) = 0$. However, *f*(0) does not exist. ✓

$\therefore$ *f* is not continuous at *x* = 0. ✓ [2 *marks*]

[total 11 *marks*]

9.(a)

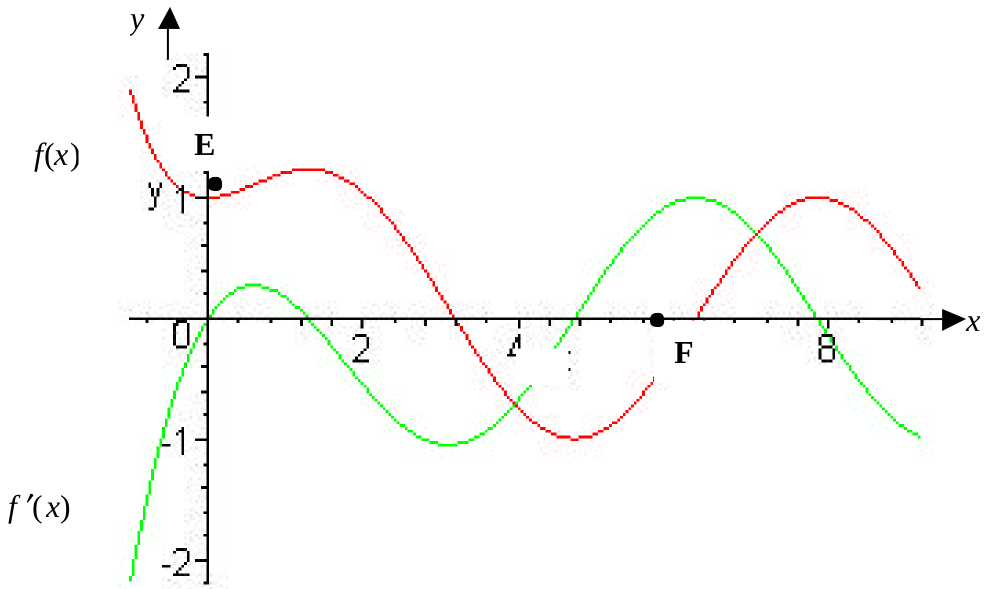
Point	f'	f''
A	-	+
B	-	0
C	0	+
D	+	-

✓✓ ✓✓

[4 *marks*]

9.(b) See graph below for E ✓ and F. ✓ [2 *marks*]

9.(c)



[5 *marks*]

9.(d) Global maximum: (-1,1.8) or (-1,1.9). ✓

Global minimum: (0,1). ✓ [2 *marks*]

[total 13 *marks*]

10.(a) $\dfrac{dV}{dt} = 30\cos(60pt).60p = 1800p\cos(60pt)$ ✓✓ [2 *marks*]

10.(b) $\dfrac{d^2V}{dt^2} = -108000p^2 \sin(60pt)$ ✓

For $\dfrac{d^2V}{dt^2} = 0 \Rightarrow \sin(60pt) = 0 \Rightarrow 60pt = 0, p, 2p, \ldots$ ✓

$\therefore t = 0, \dfrac{1}{60}, \dfrac{1}{30}, \dfrac{1}{20}, \dfrac{1}{15}, \ldots \text{sec}$ ✓

For $t = 0 \Rightarrow \dfrac{dV}{dt} = 1800p$

For $t = \dfrac{1}{60} \Rightarrow \dfrac{dV}{dt} = 1800p\cos p = -1800p$

For $t = \dfrac{1}{30} \Rightarrow \dfrac{dV}{dt} = 1800p\cos 2p = 1800p$

$\therefore$ Maximum when $t = 0, \dfrac{1}{30}, \ldots$ Then $\dfrac{dV}{dt} = 1800p$. ✓ [4 *marks*]

10.(c) $\dfrac{dV}{dt} = 0$ when

$1800p\cos(60pt) = 0 \Rightarrow \cos(60pt) = 0 \Rightarrow 60pt = \dfrac{p}{2} \Rightarrow t = \dfrac{1}{120}, \dfrac{3}{120}, \dfrac{5}{120}, \ldots \text{sec}$

✓✓

When $t = \dfrac{1}{120}$ sec, there will be a maximum voltage. ✓ [3 *marks*]

Alternatively:

When $t = \dfrac{3}{120}$ sec, there will be a minimum voltage. (✓)

NOTE: The following is not required. No marks to be allocated.

(When $t = \dfrac{1}{120}$ sec, $V = 30\sin(60p.\dfrac{1}{120}) = 30\sin\dfrac{p}{2} = 30$.

Alternatively: When $t = \dfrac{3}{120}$ sec, $V = 30\sin(60p.\dfrac{3}{120}) = 30\sin\dfrac{3p}{2} = -30$.)

[total 9 *marks*]

---

11. Let $z = a + bi$. Then:

$\dfrac{a+(b-1)i}{(a-1)+bi} . \dfrac{(a-1)-bi}{(a-1)-bi}$ ✓ $= \dfrac{a(a-1) - abi + (a-1)(b-1)i + b(b-1)}{(a-1)^2 + b^2}$ ✓

The result is purely imaginary when $\dfrac{a(a-1)+b(b-1)}{(a-1)^2+b^2} = 0$ ✓

i.e. $a^2 - a + b^2 - b = 0$

i.e. $a^2 - a + \dfrac{1}{4} + b^2 - b + \dfrac{1}{4} = \dfrac{1}{2}$

i.e. $(a - \dfrac{1}{2})^2 + (b - \dfrac{1}{2})^2 = \dfrac{1}{2}$ ✓

This equation represents a circle with centre $(\frac{1}{2},\frac{1}{2})$ ✓ and radius = $\frac{1}{\sqrt{2}}$. ✓

[total 6 *marks*]

12.(a) $\mathbf{v}(t) = (3t^2 - 2)\mathbf{i} + (2t - 3)\mathbf{j}$. ✓

Then: Speed = $|v(t)| = \sqrt{(3t^2-2)^2+(2t-3)^2}$ ✓

When $t = 3$: Speed = $|v(3)| = \sqrt{25^2+3^2} = \sqrt{625+9} = \sqrt{634} \approx 25.18$ *m/s* ✓

[3 *marks*]

Alternatively: $\mathbf{v}(t) = (3t^2 - 2)\mathbf{i} + (2t - 3)\mathbf{j}$ (✓)

$\mathbf{v}(3) = 25\mathbf{i} + 3\mathbf{j}$ (✓)

Speed at time t is $|v(3)| = \sqrt{634} \approx 25.18$ *m/s*. (✓)

12.(b) Set $|v(t)| = 0 \Rightarrow \sqrt{(3t^2-2)^2+(2t-3)^2} = 0$ ✓

i.e. $9t^4 - 12t^2 + 4 + 4t^2 - 12t + 9 = 0$

i.e. $9t^4 - 8t^2 - 12t + 13 = 0$

This equation does not have any real roots. ✓

$\therefore$ Particle never comes to a stop. ✓ [3 *marks*]

Alternatively: Particle stops when $\mathbf{v}(t) = 0$

i.e. $3t^2 - 2 = 0$ and $2t - 3 = 0$ (✓)

i.e. $t^2 = \frac{2}{3}$ and $t = \frac{3}{2}$, which is impossible. (✓)

$\therefore$ Particle never comes to a stop. (✓)

12.(c) Particle is moving horizontally when **j** component equals zero.

i.e. $2t - 3 = 0$ ✓

$\therefore t = \frac{3}{2}$ seconds. [2 *marks*]

Position vector when $t = \frac{3}{2}$ is: $\mathbf{r}(\frac{3}{2}) = 0.38\mathbf{i} - 2.25\mathbf{j}$. ✓ [total 8 *marks*]

13.(a) $\frac{1}{2}(x^2 - 5x)\sqrt{x} = -3\sqrt{x}$

i.e. $\sqrt{x}(x^2 - 5x + 6) = 0$

i.e. $x = 0, \quad x = 2, \quad x = 3$

$\therefore$ A$(2,-3\sqrt{2}) = (2,-4.24)$ and B$(3,-3\sqrt{3}) = (3, -5.20)$. ✓✓ [2 *marks*]

Alternatively: Using graphics calculator gives (2,-4.24) and (3,-5.20). (✓✓)

13.(b) Required Area = $\int_0^2 (f(x) - g(x))dx$ ✓ + $\int_2^3 (g(x) - f(x))dx$ ✓ =1.75 square units. ✓

[3 *marks*]

13.(c) S (5,0) ✓

P $(p,-3\sqrt{p})$ or P $(x, -3\sqrt{x})$ ✓

$D = \sqrt{(5-p)^2 + (-3\sqrt{p})^2} = \sqrt{p^2 - p + 25}$ ✓

$\dfrac{dD}{dp} = \dfrac{2p-1}{2\sqrt{p^2 - p + 25}}$ ✓

For $\dfrac{dD}{dp} = 0 \Rightarrow p = \dfrac{1}{2}$ ✓

When $p = 0 \Rightarrow \dfrac{dD}{dp} < 0$ and $p = 1 \Rightarrow \dfrac{dD}{dp} > 0$.

For $p = \dfrac{1}{2}$ a minimum is obtained. ✓

$\therefore$ Minimum distance $= \sqrt{(4\frac{1}{2})^2 + \frac{9}{2}} = \sqrt{\frac{81}{4} + \frac{18}{4}} = \frac{3}{2}\sqrt{11}$ units. [7 *marks*]

= 4.97 (or 5) units. ✓

Alternatively: Determine nature of turning point from graphics calculator.

[total 12 *marks*]

14.(a) $T(0) = 25$. ✓

Then: $25 = p(1 - e^{-0k}) + q$

i.e. $25 = p(1 - 1) + q$

$\therefore$ $q = 25$. ✓

For $t \to \infty$: $190 = p(1 - e^{-k\infty}) + q$ ✓

i.e. $190 = p + q$

i.e. $p = 190 - 25$

$\therefore p = 165$. ✓ [4 *marks*]

14.(b) $T?(t) = 165ke^{-kt}$ ✓

For $t = 0 \Rightarrow 165ke^0 = 3$ ✓

Then: $165\,k = 3$

$\therefore k = \dfrac{1}{55}$. ✓ [3 *marks*]

14.(c) For $t = 5 \Rightarrow T?(5) = \dfrac{165}{55}e^{-\frac{5}{55}} = 2.74^0$ C/minute. ✓✓ [2 *marks*]

[total 9 *marks*]

15.(a) 2 cis $?$ = 2 $e^{i?}$ ✓ [1 *mark*]

15.(b) $\dfrac{d}{dq}(2e^{iq})$ = 2 $i\,e^{i?}$ ✓ [1 *mark*]

15.(c) $q = \dfrac{p}{3} \Rightarrow w = 2\,i\,e^{i\frac{p}{3}} = 2\,i$ cis $\dfrac{p}{3} = 2\,i\,(\cos\dfrac{p}{3} + i\sin\dfrac{p}{3}) = 2\,i\,(\dfrac{1}{2} + \dfrac{1}{2}i\sqrt{3})$

$= -\sqrt{3} + i$. ✓✓ [2 *marks*]

15.(d) $z = 1 + i\sqrt{3}$ and $w = -\sqrt{3} + i$. From $z \to w$ as vectors we notice that they are perpendicular because the dot product equals zero. ✓✓ [2 *marks*]

[total 6 *marks*]

16.(a) $\frac{dI}{dt} = -cI \Leftrightarrow I = I_0 e^{-ct}$ ✓

Then: $\frac{1}{2}I_0 = I_0 e^{-5568c} \Rightarrow \ln\frac{1}{2} = -5568c \Rightarrow c = 0.0001245$ (7 d.p.) ✓✓

[3 *marks*]

16.(b) Now $I = 0.03I_0$. ✓

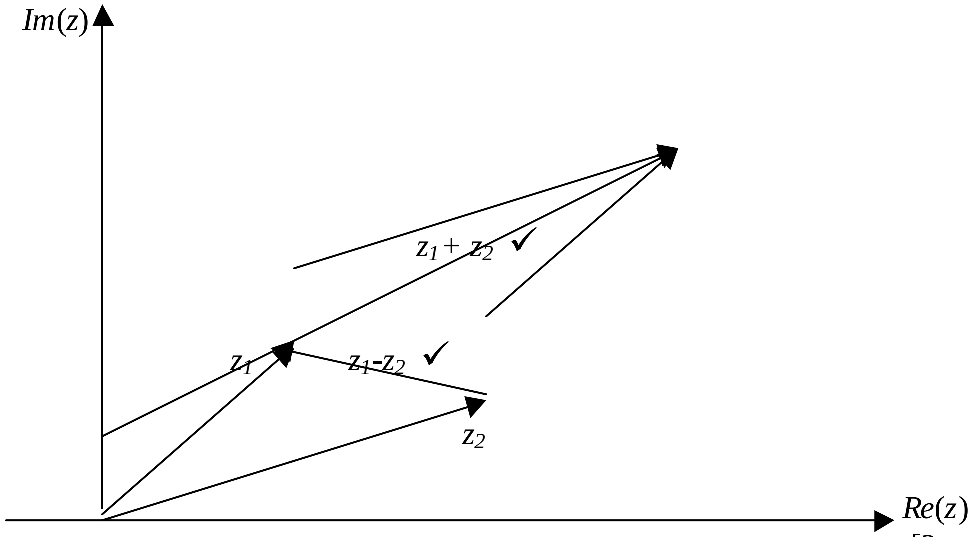
Then: $0.03 = e^{-0.0001245t}$ ✓

i.e. $\ln 0.03 = -0.0001245t$

$\therefore t = \frac{\ln 0.03}{-0.0001245}$ = 28 168 years ✓ [3 *marks*]

[total 6 *marks*]

17.(a)



[2 *marks*]

17.(b) $|z_1 + z_2| = |z_1 - z_2|$

Then: $(z_1 + z_2)^2 = (z_1 - z_2)^2$ ✓

i.e. $2z_1z_2 = -2z_1z_2$

Then: $z_1z_2 = 0$. ✓

If z_1 and z_2 are vectors then $\mathbf{z}_1.\mathbf{z}_2 = 0$ means that $\mathbf{z}_1 \perp \mathbf{z}_2$.

$\therefore$ The difference of the arguments is $\frac{p}{2}$. ✓✓ [4 *marks*]

Alternatively:

Since $|z_1 + z_2| = |z_1 - z_2|$, the parallelogram has equal diagonals. (✓) This implies that the parallelogram is a rectangle (✓) so that $\mathbf{z}_1 \perp \mathbf{z}_2$, (✓) i.e. the difference in arguments is $\frac{p}{2}$. (✓) [total 6 *marks*]

18.(a) Let $g(x) = x^n$ and $h(x) = \sin\left(\frac{px}{2}\right)$.

On the interval $0<x<1$, $g(x) > 0$. On the same interval $h(x) > 0$. ✓✓

$\therefore \int_0^1 x^n \sin(\frac{px}{2})dx > 0.$ ✓ [3 *marks*]

18.(b) On the interval 0<*x*<1, $f(x) = x^n \sin(\frac{px}{2})$ is always less than $g(x) = x^n$ for

$h(x) = \sin(\frac{px}{2}) < 1.$ ✓✓

Also: $\int_0^1 x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1} - 0 = \frac{1}{n+1}.$ ✓✓

Then: $\int_0^1 x^n \sin(\frac{px}{2})dx < \int_0^1 x^n dx = \frac{1}{n+1}$, as was required to show. [4 *marks*]

18.(c)

n	$\int_0^1 x^n \sin(\frac{px}{2})dx$	$\frac{1}{n+1}$
0	0.6366	< 1
1	0.4053	< 0.5
2	0.2945	< 0.33
3	0.2303	< 0.25
4	0.1886	< 0.2

✓✓✓✓ [4 *marks*]

ÖÖÖ

18.(d) Possible value for *p* is zero. ✓ The value for *q* should be $\frac{1}{999+1} = 0.001.$ ✓

[2 *marks*]

[total 13 *marks*]

19.(a) $y = 2x + \frac{1}{2}$

$\frac{dy}{dt} = 2\frac{dx}{dt} = 2 \text{ x } 4 = 8$ units/sec. ✓ [1 *mark*]

19.(b) $s^2 = (x-5)^2 + (y-3)^2$ ✓

Then: $2s\frac{ds}{dt} = 2(x-5).\frac{dx}{dt} + 2(y-3).\frac{dy}{dt}$ ✓

i.e. $\frac{ds}{dt} = \frac{(x-5).\frac{dx}{dt} + (y-3).\frac{dy}{dt}}{[(x-5)^2 + (y-3)^2]^{\frac{1}{2}}}$

At (-1,-1): $\frac{ds}{dt} = \frac{(-1-5).(4) + (-1-3).(8)}{[(-1-5)^2 + (-1-3)^2]^{\frac{1}{2}}} = \frac{-56}{\sqrt{52}}$ units/sec (-7.8 u/s) ✓

[3 *marks*]

19.(c) The negative implies that P is **getting closer** to Q. ✓ [1 *mark*]

19.(d) $\frac{ds}{dt} = 0$.

i.e. $(x-5).4 + (y-3).8 = 0$

i.e. $4x - 20 + 8y - 24 = 0$

i.e. $x + 2y - 11 = 0$ ✓

Since P lies on $y = 2x + \frac{1}{2}$

Then $x + 2(2x + \frac{1}{2}) - 11 = 0$

i.e. $5x = 10$

i.e. $x = 2$

$\therefore$ Point of location is (2,4.5). ✓ [2 *marks*]

[total 7 *marks*]

20.(a) $\mathbf{a}(t) = 0\mathbf{i} - 9.8\mathbf{j}$ ✓

There is only a negative vertical acceleration (due to gravity). ✓ [2 *marks*]

20.(b) $\mathbf{r}(t) = (ß \cos \frac{p}{6} t + c_1)\mathbf{i} + (ß\ sin\ \frac{p}{6} t - 4.9\, t^2 + c_2)\mathbf{j}$ ✓

At $t = 0 \Rightarrow$ A(0,100)

Then: $ß \cos \frac{p}{6} \times 0 + c_1 = 0 \Rightarrow c_1 = 0$ ✓

Also: $ß \sin \frac{p}{6} \times 0 - 4.9 \times 0 + c_2 = 100 \Rightarrow c_2 = 100$ ✓

$\therefore$ $\mathbf{r}(t) = (ß \cos \frac{p}{6})t\mathbf{i} + (ß \sin \frac{p}{6} t - 4.9\, t^2 + 100)\mathbf{j}$ ✓

i.e. $\mathbf{r}(t) = \frac{1}{2}\, ß\, t\sqrt{3}\,\mathbf{i} + (-4.9\, t^2 + \frac{1}{2}\, ßt + 100)\mathbf{j}$ [4 *marks*]

20.(c) Maximum height is obtained when $t = 30$ seconds.

Then the vertical component of the velocity vector is equal to zero.

i.e. $ß \sin\frac{p}{6} - 9.8 \times 30 = 0$ ✓

i.e. $\frac{1}{2}\, ß = 294$

$\therefore$ $ß = 588$ m/s (initial speed). ✓ [2 *marks*]

20.(d) The rocket hits the ground when the vertical component of the position vector is equal to 0 m. ✓

Then: $-4.9\, t^2 + 294t + 100 = 0$ ✓

i.e. $4.9\, t^2 - 294t - 100 = 0$

i.e. $t = 60.34$ seconds ✓

Then the horizontal component of the position vector indicates the distance from O to B.

i.e. OB = $294 \times 60.34 \times \sqrt{3} \approx 30726.5$ m ≈ 30.7 km. ✓

(Give credit for $30.5 \le OB \le 31$) [4 *marks*]

[total 12 *marks*]

21.(a) $\frac{dl}{dt} = k_1 a \Rightarrow \frac{d^2l}{dt^2} = k_1 \frac{da}{dt} \Rightarrow \frac{d^2l}{dt^2} = k_1(-k_2 l)$ ✓✓

$\therefore \frac{d^2l}{dt^2} + k_1 k_2 l = 0$ ✓ [3 *marks*]

21.(b) $\frac{d^2l}{dt^2} = k_1(-k_2 l) \Rightarrow \frac{d^2l}{dt^2} = -4l$. This is a Simple Harmonic Motion. ✓

Then: $l(t)$ = A cos (2t + ?) and $a(t)$ = -A sin (2t + ?) since $a(t) = \frac{1}{k_1}\frac{dl}{dt}$. ✓✓

Now: 250 = A cos ? ✓ and 4000 = -A sin ?. ✓

Then: tan ? = -16 $\Rightarrow q = -1.508$ (3 d.p.) and A = 4008 (nearest unit.)

$\therefore$ $l(t)$ = 4008 cos(2t – 1.508) ✓ and $a(t)$ = -4008 sin(2t – 1.508) ✓

[7 *marks*]

21.(c) Then: 0 = -4008 sin(2t – 1.508) $\Rightarrow 2t - 1.508 = 0 \Rightarrow t = 0.75$ (2 d.p.) ✓✓

$\therefore$ After 75 years the prey is extinct. ✓ [3 *marks*]

[total 13 *marks*]

97902_1.DOC